ISRAEL JOURNAL OF MATHEMATICS **134** (2003), 317–343

ENTIRE FUNCTIONS HAVING A CONCORDANT VALUE SEQUENCE

BY

JONATHAN PILA*

Department of Mathematics and Statistics, University of Melbourne, Australia
e-mail: pila@ms.unimelb.edu.au

ABSTRACT

A classic theorem of Pólya shows that 2^z is, in a strong sense, the "smallest" transcendental entire function that is integer valued on $\mathbb{N}$. An analogous result of Gel'fond concerns entire functions that are integer valued on the set $X_a = \{a^n \colon n \in \mathbb{N}\}$, where $a \in \mathbb{Z}, |a| \geq 2$. Let $X = \mathbb{N}$ or $X = X_a$ and $k \in \mathbb{N}$ or $k = \infty$. This paper pursues analogous results for entire functions f having the following property: on any finite subset D of X with $\#D \leq k+1$, the values $f(z), z \in D$ admit interpolation by an element of $\mathbb{Z}[z]$. The results obtained assert that if the growth of f is suitably restricted then the restriction of f to X must be a polynomial. When $X = X_a$ and $k < \infty$ a "smallest" transcendental entire function having the requisite property is constructed.

1. Introduction

This paper is a further contribution to the large literature stemming from the classic paper [27] of Pólya. Like many of its predecessors, it is devoted to establishing results of the following general form:

> *Let X be a suitable discrete subset of $\mathbb{C}$, and Π a suitable set of properties. Suppose that f is an entire function whose restriction to X satisfies Π. If the growth of f is suitably restricted, then f is a polynomial.*

---

* *Mail address:* 6 Goldthorns Avenue, Kew 3101, Australia.
Received November 19, 2001

Thus the original and paradigm result ([27], as sharpened by [18] and [28]): If the entire function f, with $M(f,r) = \max\{|f(z)|: |z| \le r\}$, is integer valued on $\mathbb{N} = \{0, 1, 2, \ldots\}$ and

$$\limsup_{r\to\infty} \frac{M(f,r)}{2^r} < 1,$$

then f is a polynomial.

This result is sharp in the sense that the growth condition on f is that it grows slightly slower than a known transcendental entire function $T_{X,\Pi}$ (in this case the function 2^z) that satisfies Π on X and which can accordingly lay claim to being the "smallest" such function. The result thus takes the sharper form:

> *Let X be a suitable discrete subset of $\mathbb{C}$, and Π a suitable set of properties. There is a "smallest" transcendental entire function $T_{X,\Pi}$ whose restriction to X satisfies Π.*

The main purpose of this paper is to obtain results of the above forms for $X = \mathbb{N}$ and for $X = X_a = \{a^n: n \in \mathbb{N}\}$ where $a \in \mathbb{Z}, |a| \ge 2$, with the properties Π as set out in the following definition. A secondary purpose of the paper is to study these properties rather more generally, and accordingly the definitions are given in a general setting. All rings in this paper are commutative with unit.

Definition 1.1: Let R be a ring, $Y \subset R$, $k \in \mathbb{N}$. A function $f: Y \to R$ will be called **concordant to order k on Y in R** if for any $\kappa \le k$ and any choice $y_0, y_1, \ldots, y_\kappa \in Y$ there exists $P \in R[t]$ such that $P(y_i) = f(y_i), i = 0, \ldots, \kappa$. A function f that is concordant on Y in R to every order $k \in \mathbb{N}$ will be called **superconcordant on Y in R.**

Thus, for $X \subset \mathbb{Z}$ and $f: X \to \mathbb{C}$, concordance to order 0 on X in $\mathbb{Z}$ is simply the property of being integer valued on X, as in Pólya's original result. A series of papers considers this property for other subject sets $X \subset \mathbb{Z}$. Pólya's original paper considers $X = \mathbb{Z}$ as well as $X = \mathbb{N}$. Gel'fond [12, or 13] considers X of the form X_a. Bézivin [2] obtains a result for non-periodic sequences of the form $X = X_{P,x_0} = \{x_n: n \in \mathbb{N}\}$, where $x_0 \in \mathbb{Z}, P \in \mathbb{Z}[t]$ and $x_{i+1} = P(x_i)$. Further extensions applying to suitable sparse subsets of $\mathbb{Z}$ are pursued in [4, 25]. All these results are of the sharper form (to varying degrees of sharpness) in that they exhibit a suitably smallest transcendental entire function that is integer valued on X.

The subject of integer valued entire functions has been further pursued in many other directions. For example, results of Pólya [28], Selberg [31, 32] and Pisot [26] (see also the discussion and other references in Narkiewicz [20]) characterizing entire functions that are integer valued on $\mathbb{N}$ but grow somewhat faster than 2^z; results on entire functions mapping the ring of integers of a given number field into itself [16] (again see [20] for further references); results of Gel'fond [15, Chapter III] on functions integer valued on sets with certain lattice properties; results in which the growth of f is restricted according to direction [6, 29]; integrality of f and certain of its derivatives (see references below); functions of several variables [7]; analogues in function fields [9]. References to several recent survey articles are given in [20]. Earlier developments are surveyed in [5].

Concordance to order 1 on X in $\mathbb{Z}$, for a function $f\colon X \to \mathbb{C}$ where $X \subset \mathbb{Z}$, is equivalent to having $f(X) \subset \mathbb{Z}$ and, for all $x, y \in X$,

$$f(x) - f(y) \in \mathbb{Z}.(x-y).$$

Perelli and Zannier [22] consider a slightly weaker property for $X = \mathbb{N}$, requiring $f(n+p) - f(n) \in \mathbb{Z}.p$ for $n \in \mathbb{N}$ and all sufficiently large prime numbers p. They prove that an entire function of exponential type less than $\log(e+1)$ having this property on $\mathbb{N}$ must be a polynomial, but their result is not of the sharper form: They do not construct a transcendental entire function of exponential type $\log(e+1)$ enjoying this property on $\mathbb{N}$. Bézivin [4] considers a property slightly weaker than concordance to order 1 for X_a, requiring, for $n, m \in \mathbb{N}$, that $f(a^{n+m}) - f(a^n) \in \mathbb{Z}.(a^m - 1)$. Concordance to order 1, as well as some weakenings of it, are considered for certain sparse sequences of integers in [24]. The results of [4, 24] include the construction of suitably smallest transcendental entire functions having the requisite property on their subject sequences.

In this paper the following result is established for $X = X_a$. For $k = 0$ it is (a less sharp form of) the result of Gel'fond [12], and for $k = 1$ it is closely related to the result of Bézivin [3]. For $k \in \mathbb{N}$ set $\alpha_k = (3/\pi^2)\sum_{\kappa=1}^{k} \kappa^{-2}$ (here, and throughout the paper, the empty sum is taken to have the value 0).

THEOREM 1.2: *Let $a \in \mathbb{Z}, |a| \geq 2$, $k \in \mathbb{N}$. There exists a transcendental entire function $T_{a,k}$ that is concordant to order k on X_a in $\mathbb{Z}$ and has*

$$\limsup_{r\to\infty} \frac{\log M(T_{a,k}, r)}{(\log r)^2} = \frac{1}{4(1-\alpha_k)\log|a|}.$$

Let f be an entire function that is concordant to order k on X_a in $\mathbb{Z}$ and suppose

$$\limsup_{r\to\infty} \frac{\log M(f, r)}{(\log r)^2} < \frac{1}{4(1-\alpha_k)\log|a|}.$$

Then f is a polynomial.

Before turning to the corresponding statement for $X = \mathbb{N}$ some preliminary remarks must be made. The method of proof employed here follows that of Pólya [27] and much of the subsequent work. It proceeds in two stages. In the first step a polynomial $P \in \mathbb{Q}[t]$ is constructed such that $P(x) = f(x)$ for $x \in X$. So $P - f$ vanishes on X and has controlled growth; the second step shows that in fact $P - f$ must vanish identically. Pólya uses Jensen's Theorem to effect this step; Carlson's Theorem, proved subsequently, implies that entire functions of exponential type $< \pi$ are uniquely determined by their value sequence on $\mathbb{N}$. The value π is optimal in view of the function $\sin(\pi x)$. In the following result the growth restrictions exceed the critical exponential type π (once $k > 11$) and this second step cannot be effected. Thus the result stated for all k is simply that the restriction of f to $\mathbb{N}$ is a polynomial, i.e., the value sequence $f(\mathbb{N})$ is the value sequence of a polynomial.

The case $k = 1$ of the following result follows from the result of [22]; for $k = 0$ it is a weak form of Pólya's original result. For $k \in \mathbb{N}$ let $\gamma_k = \sum_{\kappa=1}^{k} 1/\kappa$.

THEOREM 1.3: *Let $k \in \mathbb{N}$. Let f be an entire function that is concordant to order k on $\mathbb{N}$ in $\mathbb{Z}$. Suppose*

$$\limsup_{r\to\infty} \frac{\log M(f,r)}{r} < \log(e^{\gamma_k} + 1).$$

Then there exists $P \in \mathbb{Z}[t]$ with $f(n) = P(n)$ for $n \in \mathbb{N}$.

For $k \geq 1$, the existence of a transcendental entire function $T_{\mathbb{N},k}$ that is concordant to order k on $\mathbb{N}$ and of the critical exponential type $\log(e^{\gamma_k} + 1)$ seems to present an interesting problem. It will be seen that the method of proof suggests a natural candidate for the value sequence $T_{\mathbb{N},k}(n), n \in \mathbb{N}$.

A feature of concordance to any order and of superconcordance is that they depend only on the value sequence $f(X)$ of f on X unlike, for example, the requirement that higher derivatives of f be integer valued on X. The latter has been studied for $X = \mathbb{N}$ (see [11] and earlier references in [5] or [20]) and $X = X_a$ in [14, 7, 33].

A further feature of concordance to any order and of superconcordance is that they are obviously enjoyed by elements of $\mathbb{Z}[z]$ on any $X \subset \mathbb{Z}$. Thus results of the form under consideration have a *local-global* character, somewhat impaired by the fact that for $k \in \mathbb{N}$ and $X \subset \mathbb{Z}$ there are polynomials in $\mathbb{Q}[t] \backslash \mathbb{Z}[t]$ that are concordant to order k on X in $\mathbb{Z}$. If $X \subset \mathbb{Z}$ is infinite, however, the elements of $\mathbb{Q}[t]$ that are superconcordant on X in $\mathbb{Z}$ are precisely $\mathbb{Z}[t]$. Thus for the property of superconcordance a true local-global result can be anticipated.

Here the growth rates for $X = X_a$ as well as $\mathbb{N}$ are outside the range enjoying uniqueness of interpolation, so the conclusion is about the value sequence $f(X)$. Further, as in 1.3, no smallest superconcordant transcendental entire function $T_{X,\infty}$ is constructed. However a natural candidate value sequence $T_{X,\infty}(x_n), x_n \in X$ for such a function is suggested by the proof. Let $\Gamma(z)$ denote the gamma function.

THEOREM 1.4: *Let f be an entire function that is superconcordant on $\mathbb{N}$ in $\mathbb{Z}$. Suppose*

$$\limsup_{r\to\infty} \frac{M(f,r)}{\Gamma(r)} < 1.$$

Then there exists $P \in \mathbb{Z}[t]$ with $f(n) = P(n)$ for $n \in \mathbb{N}$.

THEOREM 1.5: *Let $a \in \mathbb{Z}, |a| \geq 2$. Let f be an entire function that is superconcordant on X_a in $\mathbb{Z}$. Suppose*

$$\limsup_{r\to\infty} \frac{\log M(f,r)}{(\log r)^2} < \frac{1}{\log |a|}.$$

Then there exists $P \in \mathbb{Z}[t]$ with $f(a^n) = P(a^n)$ for $n \in \mathbb{N}$.

The method of proof of the above results follows the same basic line of [27] and much of the subsequent work. The following outline of the method to be employed serves also to indicate the structure of the rest of the paper.

Definition 1.6: A sequence X in a ring R will be called **proper** if it contains no repeated terms.

Let $X = \{x_0, x_1, \ldots\} \subset \mathbb{C}$ be a proper sequence. Define polynomials $\phi_{X,n} \in \mathbb{C}[t]$ of degree $n \in \mathbb{N}$ by

$$\phi_{X,n}(t) = \frac{\prod_{i=0}^{n-1}(t - x_i)}{\prod_{i=0}^{n-1}(x_n - x_i)}$$

(here, and throughout the paper, the empty product is taken to have value 1).

Thus $\phi_{X,n}$ vanishes at $x_0, \ldots, x_{n-1}$ while $\phi_{X,n}(x_n) = 1$. It follows that any complex-valued function f defined on X can, on X, be expanded in a (Newton interpolation) series

$$f(x_n) = \sum_{j=0}^{\infty} c(j)\phi_{X,j}(x_n),$$

the coefficients $c(n)$ being (uniquely) determined inductively (see 2.8 below).

The proof of the main results will be effected by showing that, under the hypotheses made on $f(z)$, the corresponding coefficients $c(n)$ vanish once n is sufficiently large. This will be forced by the combination of an estimate of $c(n)$, involving the growth function $M(f,r)$ of f, and arithmetic information about $c(n)$ following from the assumed properties of the value sequence of f on X.

The coefficients $c(n)$ may be expressed (as follows from [25, 4.5]) by a contour integral. For $r > r_n = \max(|x_0|, \ldots, |x_n|)$, and C_r the disk of radius r,

$$c(n) = \frac{1}{2\pi i} \int_{C_r} \frac{f(z)dz}{\phi_{X,n}(z)(z - x_n)}.$$

This leads to an obvious estimate for $|c(n)|$ in terms of $M(f,r)$, appropriate formulations of which are pursued in Section 4.

The connection between concordance of f on X in $\mathbb{Z}$ and arithmetic properties of the expansion coefficients $c(n)$ is provided by the next result. This result will be established in a general setting in Sections 2 and 3 applicable to sequences having the following property.

Definition 1.7: A sequence X in a ring R is called **concordant** if

$$x_{j+m} - x_j \in R.(x_{i+n} - x_i)$$

whenever $i \leq j$ and $n|m$.

Note that a sequence X is concordant if and only if its successor function $\sigma_X : X \to X$, $\sigma_X(x_j) = x_{j+1}$ is concordant to order 1 on X in R, hence the use of the same term. The sequences $\mathbb{N}, X_a, X_{P,x_0}$ described above are all concordant in $\mathbb{Z}$, as are the subject sequences of [25], [24]. If $X \subset \mathbb{Z}$ is a proper concordant sequence then (by [25, Prop. 1.3] or 2.7 below) the polynomials $\phi_{X,n}$ have the additional property of being integer valued on X. In that case the coefficients $c(n)$ belong to the ring generated by $f(X)$. In particular, $c(n) \in \mathbb{Z}$ for all $n \in \mathbb{N}$ if f is integer valued on X.

Definition 1.8: For a sequence $X = \{x_j : j \in \mathbb{N}\} \subset \mathbb{Z}$ and $k, n \in \mathbb{N}$ denote by

$$\lambda_X(k,n)$$

the least common multiple of the set of products of any $\kappa \leq k$ distinct elements of $\{x_n - x_0, x_n - x_1, \ldots, x_n - x_{n-1}\}$, with $\lambda_X(0,n) = \lambda_X(k,0) = 1$.

For $f: \mathbb{Z} \to \mathbb{Z}$ an equivalent characterization to that given in the following result is given in Aichinger [1]. For $X = \mathbb{N}$ and $k = 1$ the result also appears in [17, 30].

THEOREM 1.9: *Let $k \in \mathbb{Z}$. Suppose $X = \{x_j, j \in \mathbb{N}\} \subset \mathbb{Z}$ is a proper concordant sequence, and $f\colon X \to \mathbb{Z}$. Let $c(j) \in \mathbb{Z}, j \in \mathbb{N}$ be chosen so that, for $n \in \mathbb{N}$,*

$$f(x_n) = \sum_{j=0}^{\infty} c(j)\phi_{X,j}(x_n).$$

Then the following statements are equivalent:

(1) *f is concordant to order k on X in $\mathbb{Z}$;*
(2) *for each $j \in \mathbb{N}$, $c(j) \in \mathbb{Z}.\lambda_X(k,j)$.*

With this characterization, and the estimates established in Section 4, the main theorems are proved in Section 5.

Finally, some remarks on directions for further work. The existence of transcendental entire functions that are concordant to order $k \geq 1$ on $\mathbb{N}$ and of the minimal exponential type admitted by Theorem 1.3 presents an interesting question, as mentioned above. Further, it would be interesting to explore analogues for higher order concordance of the above-mentioned results [28, 31, 32, 26, 29] characterizing integer valued entire functions with growth rates somewhat faster than the minimum possible exponential type.

The present methods could also be applied to other properties depending on the value sequence $f(X)$. For example, the property of having integer divided differences up to a given order. For polynomials in $\mathbb{Q}[t]$ this property is discussed in [20]. Concordance properties of higher order could be investigated in the context of arithmetical functions on a sequence X, as has been studied, for properties similar to concordance to order 1, on $X = \mathbb{N}$ in [17, 22, 23, 30, 34], and proposed in [4] for $X = X_a$.

ACKNOWLEDGEMENT: I wish to thank Hendrik Lenstra for a timely correction. I am also grateful to the referee for comments.

2. Concordant sequences

The goal of this and the next section is a generalization of Theorem 1.9 applicable when X is a concordant sequence in a domain R.

In this section it is shown that, for a concordant sequence $X = \{x_j, j \in \mathbb{N}\}$ in a ring R, finite sets of elements of the form $x_i - x_j$ admit "greatest common divisors" that will play the role played by the elements $\lambda_X(k,n)$ in Theorem 1.9. Suitable expressions for such elements are obtained that will be used for the integer sequences X_a to estimate the size of $\lambda_X(k,n)$. It is further shown that, in a domain, the Newton interpolation poynomials $\phi_{X,n}$ defined as in Section 1 have the property that they are R-valued on X, permitting any $f: X \to R$ to be expanded in a series over X.

The following key result concerning concordant sequences is due to Hendrik Lenstra. It is proved in [24, Theorem 2.1]. Let Ω be the partially ordered set consisting of pairs (i, n) where $i, n \in \mathbb{N}, n > 0$, the order being that $(i, n) \le (j, m)$ if and only if $i \le j$ and $n | m$.

THEOREM 2.1: *Let X be a concordant sequence in a ring R. Then there are elements $\sigma_{i,n} \in R$ for $(i, n) \in \Omega$ such that, for all $(j, m) \in \Omega$,*

$$x_{j+m} - x_j = \prod_{(i,n)\le(j,m)} \sigma_{i,n}.$$

Let P be a partially ordered set satisfying the following two conditions:

(a) for each $p \in P$, the number $u(p) = \#\{q \in P: q \le p\}$ is finite;

(b) any two elements $p, q \in P$ have a *greatest lower bound* $r \in P$, that is, r satisfies that, for any $s \in P$, $s \le p$ and $s \le q$ if and only if $s \le r$.

The element r in (b) is uniquely determined by p and q; it will be denoted $p \wedge q$.

If X is concordant then, as shown in [24, 2.4],

$$R.(x_{i+n} - x_i) + R.(x_{j+m} - x_j) = R.(x_{h+k} - x_h)$$

where $h = \min(i, j), k = \gcd(n, m)$. Thus the following result, which is a generalization of [24, 2.6], may be applied with the choices $P = \Omega$, $(i, n) \wedge (j, m) = (\min(i, j), \gcd(n, m))$, $z_{i,n} = x_{i+n} - x_i$ and $\sigma_{i,n}$ as provided by 2.1. The proof follows the proof of [24, 2.6] very closely.

PROPOSITION 2.2: *Let P be a partially ordered set satisfying* (a) *and* (b) *and let R be a ring. Let $\{z_p: p \in P\}$ be a family of elements of R with the property that*

$$R.z_p + R.z_q = R.z_{p\wedge q}$$

for all $p, q \in P$, and let $\{\sigma_p: p \in P\}$ be a family of elements of R with the property that

$$z_p = \prod_{q\le p} \sigma_q$$

for all $p \in P$. For a finite subset $B \subset P$ and $k \in \mathbb{N}, k \ge 1$ define

$$\beta(k, B) = \prod_{r} \sigma_r,$$

the product ranging over $\{r \in P: \#\{q \in B: r \le q\} \ge k\}$ and, for $k \in \mathbb{N}$,

$$b(k, B) = \prod_{\kappa\le k} \beta(\kappa, B).$$

Then

$$\bigcap_D R.\prod_{p\in D} z_p = R.b(k,B),$$

where the intersection extends over subsets D of B with $\#D \le k$ (for $k = 0$ the empty intersection is taken to be R, the empty product $b(0,B) = 1 \in R$. Likewise, for the empty set $\{\}$, $\beta(k,\{\}) = 1$).

Proof: Consider first the inclusion $\supset$. To see that $b(k,B)$ belongs to each of the ideals $R.\prod_{p\in D} z_p$, express all quantities in terms of σ_r. Let $r \in P$. The exponent of σ_r in $b(k,B)$ is $\max(k, \#\{q \in B\colon r \le q\})$, which is clearly at least its exponent $\#\{q \in D\colon r \le q\}$ in $\prod_{p\in D} z_p$.

To prove the other inclusion, take $z \in \cap_D R.\prod_{p\in D} z_p$. It must be shown that $z \in R.b(B,k)$.

Consider first the case in which R is a local ring with maximal ideal M. Elements of $R\backslash M$ are units. For $\kappa = 1,\dots,k$ set $S_\kappa = \{r \in P, \sigma_r \in M, \#\{q \in B, r \le q\} \ge \kappa\}$. If S_1 is empty then the elements $\beta(k,B)$ are all units and the conclusion $z \in R.b(k,B)$ is obvious. So it may be supposed that S_1 is nonempty.

Distinguish now two cases. First suppose that S_1 is a totally ordered subset of P. Then each S_κ is totally ordered, and, if nonempty, has a unique maximal element s_κ. Let $h = \min(k, \max(\kappa, S_\kappa \ne \{\}))$. Choose $q_1, \dots q_h \in B$ distinct as follows: $s_1 \le q_1; s_2 \le q_2 \notin \{q_1\}; \dots; s_h \le q_h \notin \{q_1,\dots,q_{h-1}\}$. Then $\beta(\kappa,B)$ is a unit multiple of z_{q_κ} for $1 \le \kappa \le h$, while $\beta(\kappa,B)$ is a unit if $\kappa > h$. So $R.b(k,B) = R.z_{q_1}\dots z_{q_h}$. Taking $D = \{q_1,\dots,q_h\}$ shows that $z \in R.\prod_{p\in D} z_p = R.b(k,B)$ as required.

In the second case, S_1 is not totally ordered, so there exist q, $r \in S_1$ such that $\sigma_q, \sigma_r \in M$, and q, r are not comparable. If $s = q \wedge r$ then $s < q$, $s < r$. Writing the elements z_q, z_r, z_s as products of the appropriate elements σ_t shows that, since $\sigma_q, \sigma_r \in M$, $z_q, z_r \in M.z_s$. But then $z_s \in R.z_s = R.z_q + R.z_r \subset M.z_s$, so that there exists $m \in M$ such that $z_s = mz_s$. Then $(1-m)z_s = 0$, and since $(1-m)$ is a unit it follows that $z_s = 0$. Take $q \in B$ with $s \le q$. Then $z_q = 0$, so also $z = 0$. This proves the proposition for local R.

Consider now the general case. For each maximal ideal M of R, the image of z in the localization R_M is in the R_M-ideal generated by the image of $b(B,k)$. This means that, for some $w \notin M$, it holds that $wz \in R.b(k,B)$. Hence the ideal consisting of all such $w \in R$ with $wz \in R.b(k,B)$ is not contained in any maximal ideal of R. Therefore this ideal is the unit ideal, and contains 1. This proves the proposition. ∎

Let X be a concordant sequence in a ring R. As already observed, Proposition 2.2 may be applied to the partial order Ω, the family of elements $\{x_{i+n} - x_i, (i,n) \in \Omega\}$ and a family of elements $\sigma_{i,n}, (i,n) \in \Omega$ provided by 2.1. The next result gives expressions for the elements

$$\beta_X(k,n) = \beta_X(k, B(n)), b_X(k,n) = b_X(k, B(n))$$

provided by 2.2 where $k \in \mathbb{N}$ and, for $n \in \mathbb{N}$,

$$B(n) = \{(i, n-i) \colon i = 0, \ldots, n-1\} \subset \Omega.$$

To this end, define, for $(i,r) \in \Omega$,

$$\tau_{i,r} = \prod_{j \le i} \sigma_{j,r}.$$

Proposition 2.3: *With the above notation and assumptions*

$$\beta_X(k,n) = \prod_{r \le [n/k]} \tau_{n-kr,r},$$

$$b_X(k,n) = \prod_{\kappa \le k} \beta_X(\kappa, n) = \prod_{\kappa \le k} \prod_{r \le [n/\kappa]} \tau_{n-kr,r}.$$

Proof: The element $\sigma_{i,r}$ appears in $\beta_X(k,n)$ if there are at least k elements $x_n - x_j$ of $B(n)$ with $i \le j$ and $r | n - j$. This occurs precisely if $kr \le n$ and $i \le n - kr$. Thus

$$\beta_X(k,n) = \prod_{r \le [n/k]} \prod_{i \le n-kr} \sigma_{i,r} = \prod_{r \le [n/k]} \tau_{n-kr,r}.$$

The expression for $b_X(k,n)$ is immediate from the definition. ∎

In the case of a proper concordant sequence X in a domain, the elements $\sigma_{i,n}$ are uniquely determined, and the quantities $\tau_{i,r}$, $b_X(k,n)$ may be expressed directly in terms of the quantities $x_{i+n} - x_i$.

Theorem 2.4: *Let X be a proper concordant sequence in a domain R, and $k, n \in \mathbb{N}$. Then, for $(i,r) \in \Omega$,*

$$\tau_{i,r} = \prod_{s | r} (x_{i+r/s} - x_i)^{\mu(s)}$$

whence

$$b_X(k,n) = \prod_{\kappa \le k} \prod_{r \le [n/\kappa]} \prod_{s | r} (x_{n-\kappa r+r/s} - x_{n-\kappa r})^{\mu(s)}.$$

Proof: The expression for $\tau_{i,r}$ may be verified by evaluation, as is done in [24].

$$\prod_{s|r}(x_{i+r/s}-x_i)^{\mu(s)}=\prod_{s|r}\prod_{j\le i}\prod_{m|s/r}\sigma_{j,m}^{\mu(s)}=\prod_{j\le i}\prod_{m|n}\prod_{s}\sigma_{j,m}^{\mu(s)}$$

where the last product is over $s|n/m$. However, $\sum\mu(s)$ over $s|q$ vanishes unless $q=1$ in which case $\sum\mu(s)=1$. Hence

$$\prod_{s|r}(x_{i+r/s}-x_i)^{\mu(s)}=\prod_{j\le i}\sigma_{j,r}=\tau_{i,r}.$$

The formula for $b_X(k,n)$ now follows immediately. ∎

COROLLARY 2.5: *Let $X\subset\mathbb{Z}$ be a proper concordant sequence, $k,n\in\mathbb{N}$. Set*

$$b_X(k,n)=\prod_{\kappa=1}^{k}\prod_{r=1}^{[n/\kappa]}\prod_{s|r}(x_{n-\kappa r+r/s}-x_{n-\kappa r})^{\mu(s)}$$

(the cases $n=0$, $k=0$ are allowed). Then

$$|b_X(k,n)|=\lambda_X(k,n).$$

Proof: This follows from 2.4. ∎

For $h,j\in\mathbb{N}$ with $h\le j$ define the following polynomials,

$$\pi_{h,j}(t)=\pi_{h,j}^X(t)=\prod_{i=h}^{j-1}(t-x_i)\in R[t]$$

(the empty product is here as everywhere taken to be 1).

LEMMA 2.6: *With these definitions, suppose $n\ge j$ and express each multiplicand in $\pi_{h,j}(x_n)$ in terms of the elements $\sigma_{i,r}$. Then the exponent $e_{i,r}(\pi_{h,j}(x_n))$ with which $\sigma_{i,r}$ appears in the expression for $\pi_{h,j}(x_n)$ is given by*

$$e_{i,r}(\pi_{h,j}(x_n))=\begin{cases}\left[\frac{n-h}{r}\right]-\left[\frac{n-j}{r}\right], & \text{if } i<h;\\ \left[\frac{n-i}{r}\right]-\left[\frac{n-j}{r}\right], & \text{if } h\le i<j;\\ 0, & \text{if } j\le i.\end{cases}$$

Proof: Observe that

$$e_{i,r}(\pi_{h,j}(x_n))=\#\{g\in[h,j-1]\colon g\ge i \text{ and } r|n-g\}.$$

If $h>i$ then the condition $g\ge i$ is redundant, and the number of $\#\{g\in[h,j-1]\colon r|n-g\}$ is equal to $\#\{g\in[n-j+1,n-h]\colon r|g\}$, which is equal to the asserted expression. If $h\le i<j$ then $e_{i,r}(\pi_{h,j}(x_n))$ is equal to

$$\#\{g\in[i,j-1]\colon r|n-g\}=\#\{g\in[n-j+1,n-i]\colon r|g\}.$$

If $i \geq j$ then $\#\{g \in [h, j-1]\colon g \geq i \text{ and } r|n-g\}$ is empty. ∎

For a proper sequence X in a domain R and $n \in \mathbb{N}$ set

$$\phi_{X,n}(t) = \frac{\pi_{0,n}(t)}{\pi_{0,n}(x_n)} \in K[t].$$

COROLLARY 2.7: *Let X be a proper concordant sequence in a domain R. Then for all $j, n \in \mathbb{N}$, $\phi_{X,j}(x_n) \in R$.*

Proof: If $n < j$ then $\phi_{X,j}(x_n) = 0$, while $\phi_{X,j}(x_j) = 1$. If $n > j$, express numerator and denominator in terms of the elements $\sigma_{i,r}$. It may be assumed that $i < j$. In the denominator

$$e_{i,r}(\pi_{0,j}(x_j)) = \left[\frac{j-i}{r}\right],$$

while in the numerator

$$e_{i,r}(\pi_{0,j}(x_n)) = \left[\frac{n-i}{r}\right] - \left[\frac{n-j}{r}\right] \geq \left[\frac{j-i}{r}\right]. \quad ∎$$

PROPOSITION 2.8: *Let X be a proper concordant sequence in a domain R and suppose $f\colon X \to R$. Then there exist unique $c(n) \in R, n \in \mathbb{N}$ such that f may be expanded over X in the form*

$$f(x_n) = \sum_{j=0}^{\infty} c(j)\phi_{X,j}(x_n).$$

Proof: Set $c(x_0) = f(x_0)$ and, $c(j), j < n$ having been determined, set $c(x_n) = f(x_n) - \sum_{j=0}^{n-1} c(j)\phi_{X,j}(x_n)$. ∎

3. Concordant functions

The goal of this section is to prove Theorem 3.5, generalizing 1.9.

The notion of a function herein termed concordant to order 1 was termed simply concordant in [25]. Variant notions have been variously named. In [10], a function $f\colon \mathbb{Z} \to \mathbb{Z}$ is called **modular** if it is, in the present terminology, concordant to order 1 on $\mathbb{Z}$; while such a function defined on $\mathbb{N}_{>0}$ is there termed **universal**, in [17] such a function is called a **pseudopolynomial**, in [30]: *congruence preserving*. In [21], a function $f\colon R \to R$ defined and concordant on a ring R is called **compatible**. See the discussion in [20]. A function that is superconcordant on $\mathbb{Z}$ is termed **local polynomial** in [1].

Let R be a ring and $Y \subset R$. Any $f\colon Y \to R$ is concordant to order 0 on X in R. It is further almost immediate from the definition that $f\colon Y \to R$ is concordant to order 1 if and only if, for each $x, y \in Y$, $f(x) - f(y) \in R.(x-y)$. The first result of this section is a generalization of this observation for higher order concordance valid in a domain.

It is convenient to introduce the following notation. For a ring R, $y_0, \ldots, y_\kappa \in R$, and $f\colon \{y_0, \ldots, y_\kappa\} \to R$ set

$$\Delta(y_0, \ldots, y_\kappa; f(y)) = \det \begin{pmatrix} 1 & y_0 & \cdots & y_0^{\kappa-1} & f(y_0) \\ 1 & y_1 & \cdots & y_1^{\kappa-1} & f(y_1) \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & y_\kappa & \cdots & y_\kappa^{\kappa-1} & f(y_\kappa) \end{pmatrix}.$$

Let also $V(y_0, y_1, \ldots, y_\kappa) = \prod_{i<j}(y_j - y_i)$ denote the Vandermonde determinant.

PROPOSITION 3.1: *Let R be a domain, $Y \subset R$, $f\colon Y \to R$, $k \in \mathbb{N}$. Then f is concordant to order k on Y in R if and only if, for every $\kappa \le k$ and $\{y_0, y_1, \ldots, y_\kappa\} \subset Y$,*

$$\Delta(y_0, y_1, \ldots, y_\kappa; f(y)) \in R.V(y_0, y_1, \ldots, y_\kappa).$$

Proof: If f is concordant to order k then, given $\kappa \le k$ and $\{y_0, y_1, \ldots, y_\kappa\}$, there exists $P \in \mathbb{Z}[t]$ with $P(y_i) = f(y_i), i = 0, \ldots, \kappa$. Thus replacing f by P in the determinant does not change it, but shows immediately that it belongs to $R.V(y_0, y_1, \ldots, y_\kappa)$. This proves the "only if" assertion.

For the "if" assertion, suppose f satisfies the determinant condition, let $\kappa \le k$ and $\{y_0, y_1, \ldots, y_\kappa\}$ be given. If $\kappa = 0$, the determinant condition reduces to $f(x_0) \in R$, whence f is concordant to order 0. Thus, proceeding by induction, it may be supposed that f is concordant to order $\kappa - 1$, and there is a polynomial $P \in \mathbb{Z}[t]$ with $P(y_i) = f(y_i), i = 0, \ldots \kappa - 1$. To complete the induction and conclude the proof it suffices to find $Q \in \mathbb{Z}[t]$ such that $Q(y_i) = 0, i = 0, \ldots, \kappa - 1$ and $Q(y_\kappa) = f(y_\kappa) - P(y_\kappa)$. Now such Q takes the form $z \prod_{i=0}^{\kappa-1}(t - x_i)$ where $z \in R$, and will have the desired property if $z \prod_{i=0}^{\kappa-1}(y_\kappa - y_i) = f(y_\kappa) - P(y_\kappa)$. Thus it remains to show that

$$f(y_\kappa) - P(y_\kappa) \in R. \prod_{i=0}^{\kappa-1}(y_\kappa - y_i).$$

Under the "if" assumption

$$\Delta(y_0, y_1, \ldots, y_\kappa; f(y) - P(y)) \in R.V(y_0, y_1, \ldots, y_\kappa).$$

But the top κ entries in the last column of this determinant vanish, so that expanding the determinant down the last column finds

$$\Delta(y_0, y_1, \ldots, y_\kappa; f(y)) = (f(y_\kappa) - P(y_\kappa))V(y_0, \ldots, y_{\kappa-1}).$$

Thus the assumptions imply

$$(f(y_\kappa) - P(y_\kappa))V(y_0, \ldots, y_{\kappa-1}) \in R.V(y_0, y_1, \ldots, y_\kappa),$$

and cancelling $V(y_0, \ldots, y_{\kappa-1})$ yields the required conclusion. ∎

LEMMA 3.2: *Let $\{z_0, z_1, \ldots\}$ be a sequence of indeterminates. For $h, n \in \mathbb{N}$ with $h < n$ set*

$$\Pi_{h,n}(t) = (t - z_h) \ldots (t - z_{n-1}) \in \mathbb{Z}[z_0, z_1, \ldots][t].$$

Let $y_0, \ldots, y_\kappa, \kappa \in \mathbb{N}$ be indeterminates. Then

$$\frac{\Delta(y_0, \ldots, y_\kappa; \pi_{h,n}(y))}{V(y_0, \ldots, y_\kappa)} = \sum_{h-1=j_0<j_1<\ldots<j_{m+1}=n} \prod_{i=0}^{\kappa} \Pi_{j_i+1, j_{i+1}}(y_i).$$

Proof: The proof will be by induction on κ. The case $\kappa = 0$ is immediate, however the induction step from $\kappa - 1$ to κ requires the assertion for $\kappa = 1$. It may be assumed that $h = 0$ without loss of generality, and the assertion for $\kappa = 1$ may be restated:

$$\Pi_{0,n}(x) - \Pi_{0,n}(y) = (x - y) \sum_{j=0}^{n-1} \Pi_{0,j}(x)\Pi_{j+1,n}(y).$$

This is proved by induction on n. For $n = 0$ it is trivial. Supposing the assertion holds for $n - 1$,

$$\begin{aligned}\Pi_{0,n}(x) - \Pi_{0,n}(y) =& (y - x_{n-1})(\Pi_{0,n-1}(x) - \Pi_{0,n-1}(y)) + (x - y)\Pi_{0,n-1}(x) \\ =& (x - y) \sum_{j=0}^{n-2} \Pi_{0,j}(x)(y - x_{n-1})\Pi_{j+1,n-1}(y) \\ &+ (x - y)\Pi_{0,n-1}(x).\end{aligned}$$

However, $(y - x_{n-1})\Pi_{j+1,n-1}(y) = \Pi_{j+1,n}(y)$. This establishes the assertion for $\kappa = 1$.

Consider now, for any function $f(y)$, the following row and column operations on $\Delta(y_0, \ldots, y_\kappa; f(y))$. Subtracting the last row from all the other rows, dividing through row $i+1$ by $y_i - y_\kappa$, and then applying suitable column operations shows that

$$\Delta(y_0,\ldots,y_\kappa;f(y)) = (-1)^\kappa \prod_{i=0}^{\kappa-1}(y_i-y_\kappa)\Delta\Big(y_0,\ldots,y_{\kappa-1};\frac{f(y)-f(y_\kappa)}{y-y_\kappa}\Big).$$

Suppose then that the conclusion of the proposition holds for $\kappa-1$ where $\kappa \geq 2$ and consider

$$\frac{\Delta(y_0,\ldots,y_\kappa;\pi_{h,n}(y))}{V(y_0,\ldots,y_\kappa)}$$
$$= \frac{(-1)^\kappa\prod_{i=0}^{\kappa-1}(y_i-y_\kappa)}{V(y_0,\ldots,y_\kappa)}\Delta\Big(y_0,\ldots,y_{\kappa-1};\frac{\pi_{h,n}(y)-\pi_{h,n}(y_\kappa)}{y-y_\kappa}\Big)$$
$$= \frac{1}{V(y_0,\ldots,y_{\kappa-1})}\sum_{h-1=j_0<j_\kappa<j_{\kappa+1}=n}\Pi_{j_\kappa+1,j_{\kappa+1}}(y_\kappa)\Delta(y_0,\ldots,y_{\kappa-1};\Pi_{h,j_\kappa}(y))$$

employing the assertion for $\kappa = 1$. Now applying the induction hypothesis

$$= \sum_{h-1=j_0<j_\kappa<j_{\kappa+1}=n}\Pi_{j_\kappa+1,j_{\kappa+1}}(y_\kappa)\sum_{h-1=j_0<j_1<\ldots<j_\kappa}\prod_{\mu=0}^{\kappa-1}\Pi_{j_\mu+1,j_{\mu+1}}(y_\mu)$$

which is readily seen to be equal to the required expression. ∎

Lemma 3.3: *Let $k,n \in \mathbb{N}$ and let $X = \{x_n, n \in \mathbb{N}\}$ be a proper concordant sequence in a domain R. Suppose $f\colon X \to R$ has $f(x_i) = 0, i = 0,\ldots,n-1$. If f is concordant to order k on X in R then $f(x_n) \in R.b_X(k,n)$.*

Proof: Let $\kappa \leq k$ and $\xi_1,\ldots,\xi_\kappa \in X\backslash\{x_n\}$ be distinct. Then there is a polynomial $P \in \mathbb{Z}[t]$ such that $P(\xi_i) = f(\xi_i) = 0, P(x_n) = f(x_n)$. Thus P has the form $P(t) = Q(t)\prod_{i=1}^n(t-\xi_i)$, where $Q \in \mathbb{Z}[t]$, and so $f(x_n) = P(x_n) = Q(x_n)\prod_{i=1}^n(x_n-\xi_i)$. Thus $f(x_n) \in R.\prod_{x\in D}(x_n - x)$ for any set D consisting of $\kappa \leq k$ distinct elements of $\{x_n - x_0, \ldots x_n - x_{n-1}\}$. Thus $f(x_n) \in R.b_X(k,n)$ by 2.2. ∎

Proposition 3.4: *Let $k,n \in \mathbb{N}$ and let X be a proper concordant sequence in a domain R. Then the function*

$$b_X(k,n)\phi_{X,n}$$

is concordant to order k on X in R.

Proof: The proof will be by application of the equivalence established in 3.1. Let $\kappa \leq k$ and $\{y_0,y_1,\ldots,y_\kappa\} \in X$, $y_i = x_{n_i}$ with $0 \leq n_1 < n_2 < \cdots < n_\kappa$. It must be shown that

$$\Delta(y_0,\ldots,y_\kappa;b_X(k,n)\phi_{X,n}(y)) \in R.V(y_0,y_1,\ldots,y_\kappa).$$

It may be assumed that $n_\kappa \geq n$, otherwise every $\phi_{X,n}(y_i) = 0$ so that the determinant vanishes and the conclusion is immediate. Now by 3.2,

$$\Delta(y_0,\ldots,y_\kappa;b_X(k,n)\phi_{X,n}(y))\Big/V(y_0,y_1,\ldots,y_\kappa)$$
$$= \frac{b_X(k,n)}{\pi_{0,n}(x_n)} \sum_{-1=j_0<j_1<j_2<\ldots<j_\kappa<j_{\kappa+1}=n-1} \prod_{i=0}^{\kappa} \pi_{j_i+1,j_{i+1}}(y_i).$$

It will be shown that each summand

$$t_{j_1,\ldots,j_\kappa} = \frac{b_X(k,n)}{\pi_{0,n}(x_n)} \prod_{i=0}^{\kappa} \pi_{j_i+1,j_{i+1}}(y_i) \in R.$$

It may be assumed that $n_i \geq j_i$ for each i, otherwise $t_{j_1,\ldots,j_m} = 0 \in R$.

To show that $t_{j_1,\ldots,j_\kappa}$ belongs to R, express each multiplicand in terms of the elements $\sigma_{i,r}$ of 2.1 and consider, for each $(i,r) \in \Omega$, the exponent of $\sigma_{i,r}$ appearing. It may be assumed that $i+r \leq n$, otherwise $\sigma_{i,r}$ does not appear in $\pi_{0,n}(x_n)$, so that its exponent in $t_{j_1,\ldots,j_\kappa}$ is obviously nonnegative.

There is a unique m such that $i \in [j_m+1, j_{m+1}]$, so that $0 \leq m \leq \kappa$. With these assumptions, the exponent of $\sigma_{i,r}$ in $t_{j_1,\ldots,j_\kappa}$ is, by 2.7,

$$\min\left(\left[\frac{n-i}{r}\right],k\right)+\left[\frac{j_{m+1}-i}{r}\right]+\sum_{\mu=m+1}^{\kappa}\left(\left[\frac{n_i-j_\mu-1}{r}\right]-\left[\frac{n_i-j_{\mu+1}}{r}\right]\right)-\left[\frac{n-i}{r}\right]$$

$$\geq \min\left(\left[\frac{n-i}{r}\right],k\right)+\left[\frac{j_{m+1}-i}{r}\right]+\sum_{\mu=m+1}^{\kappa}\left(\left[\frac{j_{\mu+1}-j_\mu-1}{r}\right]\right)-\left[\frac{n-i}{r}\right].$$

Now using repeatedly that, for any $a,b,r \in \mathbb{N}$, $[a/r]+[(b-1)/r] \geq [(a+b)/r]-1$,

$$\left[\frac{j_{m+1}-i}{r}\right]+\sum_{\mu=m+1}^{\kappa}\left(\left[\frac{j_{\mu+1}-j_\mu-1}{r}\right]\right) \geq \left[\frac{n-i}{r}\right]-(\kappa-m).$$

The left hand side above is also nonnegative. Thus the exponent of $\sigma_{i,r}$ in $t_{j_1,\ldots,j_\kappa}$ is at least

$$\min\left(\left[\frac{n-i}{r}\right],k\right)+\max\left(\left[\frac{n-i}{r}\right]-(\kappa-m),0\right)-\left[\frac{n-i}{r}\right] \geq 0.$$

This completes the proof. ∎

THEOREM 3.5: *Let X be a proper concordant sequence in a domain R, $k \in \mathbb{N}$, and $f: X \to R$. Expand*

$$f(x_n) = \sum_{j=0}^{n} c(n)\phi_{X,j}(x_n).$$

Then f is concordant to order k on X in R if an only if, for each n, $c(n) \in R.b_X(k,n)$.

Proof: The "if" assertion follows directly from 3.4, as a sum of functions concordant to order k is itself concordant to order k. The "only if" assertion will be established by induction. Certainly, if f is concordant to order k for any k then $c(0) \in R = R.b_X(k,0)$. Now suppose that $c(j) \in R.b_X(k,j)$ has been established for $j = 0,\ldots,n-1$. Then $f - \sum_{j=0}^{n-1} c(j)\phi_{X,j}$ is concordant to order k on X and vanishes at $x_0,\ldots,x_{n-1}$. Then by 3.3, $c(n) = f(x_n) - \sum_{j=0}^{n-1} c(j)\phi_{X,j}(x_n) \in R.b_X(k,n)$. This completes the proof. ∎

Proof of Theorem 1.9: Immediate from 3.5. ∎

Let X be a proper concordant sequence in a domain R with quotient field K. Theorem 3.5 provides a description of the elements of $K[t]$ that are concordant to order k on X in R.

4. Growth estimates

This section collects results connecting the growth of an entire function with the growth of the coefficients in its expansion over $X = \mathbb{N}$ or $X = X_a$. They are all rather straightforward.

The contour expression yields the following estimate, in which $m(g,r)$ is the minimum modulus of an entire function g for $|z| = r$,

LEMMA 4.1: *Let $X = \{x_j, j \in \mathbb{N}\} \subset \mathbb{C}$ be a proper sequence, and for $n \in \mathbb{N}$ set $r_n = \max(|x_0|,\ldots,|x_n|)$. Let f be an entire function and $c(n), n \in \mathbb{N}$ the coefficients in the expansion of f over X. Then*

$$|c(n)| \leq \inf_{r>r_n} \frac{r}{r-r_n} \frac{M(f,r)}{m(\phi_{X,n},r)}.$$

The following result for $X = \mathbb{N}$ is a generalization of [27, 22].

LEMMA 4.2: *Let f be an entire function with*

$$\limsup_{r\to\infty}\frac{\log M(f,r)}{r}\le b.$$

Let $c(n)$ be the coefficients in the expansion of f over $\mathbb{N}$. Then

$$\limsup_{n\to\infty}\frac{\log|c(n)|}{n}\le\log(e^b-1).$$

Proof: Let $\beta>b$, and choose $a\in(b,\beta)$. For a suitable constant C it holds that $M(f,r)\le Ce^{ar}$ for all positive r, and so, if $n\in\mathbb{N}$ and $r>n$ sufficiently large,

$$|c(n)|\le\frac{Cn!e^{ar}}{r(r-1)\ldots(r-n)}.$$

It follows that

$$\begin{aligned}|c(n)|e^{-n\log(e^\beta-1)}&\le\frac{C\Gamma(n+1)\Gamma(r-n)e^{ar-n\log(e^\beta-1)}}{\Gamma(r)}\\&\le\frac{n^n(r-n)^{r-n}}{r^r}e^{\beta r-n\log(e^\beta-1)}\end{aligned}$$

provided n and $r-n$ are sufficiently large, by Stirling's formula. Put $r=n/\xi$ where $\xi\in(0,1)$. Then, for sufficiently large n,

$$|c(n)|e^{-n\log(e^\beta-1)}\le\left(\xi^\xi(1-\xi)^{1-\xi}e^{\beta-\xi\log(e^\beta-1)}\right)^r=\left(\psi(\xi)\right)^r.$$

Now on the interval $\xi\in[0,1]$ the function $\psi(\xi)$ takes its minimum value $\psi(\xi)=1$ at $\xi=(e^\beta-1)/e^\beta$. Hence, with this choice of ξ, for sufficiently large n,

$$|c(n)|\le e^{n\log(e^\beta-1)}$$

so that

$$\limsup_{n\to\infty}\frac{\log|c(n)|}{n}\le\log(e^\beta-1).$$

Since this holds for arbitrary $\beta>b$, the conclusion follows. ∎

Throughout the remainder of this section let $a\in\mathbb{Z}, |a|\ge 2$, $X=X_a$. The following results are variants of results of [12, 13, 14, 4].

LEMMA 4.3: *Let $\beta\in[0,\infty)$ and suppose $f(z)$ is an entire function with*

$$\limsup_{r\to\infty}\frac{\log M(f,r)}{(\log r)^2}\le\frac{\beta}{\log|a|}.$$

Let $c(n)$ denote the coefficients in the expansion of f over X. Then

$$\limsup_{n\to\infty} \frac{\log |c(n)|}{n^2} \le \beta \log |a|.$$

Proof: The estimate for $|c(n)|$ given in 4.1 implies that, for suitable constant C, if $r \ge 2|a|^n$ then

$$|c(n)| \le \frac{r}{r-|a|^n} M(f,r) \frac{|a^n-1|\cdots|a^n-a^{n-1}|}{(r-1)\cdots(r-|a|^{n-1})} \le CM(f,r)|a|^{n^2} r^{-n}.$$

Let b be chosen with $\beta < b$, and set $r = 2|a|^n$. Then

$$\frac{\log M(f,r)}{(\log r)^2} \le \frac{b}{\log |a|}$$

once n is sufficiently large. Now

$$\frac{b(\log r)^2}{\log |a|} = bn^2 + O(n),$$

$$|c(n)| \le C|a|^{bn^2+O(n)}.$$

Therefore

$$\limsup_{n\to\infty} \frac{\log |c(n)|}{n^2} \le b \log |a|.$$

Since this holds for any $b > \beta$, the conclusion follows. ∎

The above may be improved over part of the growth range by a more judicious application of 4.1.

LEMMA 4.4: *Let $\beta \in (-\infty, 1/2)$ and suppose $f(z)$ is an entire function with*

$$\limsup_{r\to\infty} \frac{\log M(f,r)}{(\log r)^2} \le \frac{1}{4(1-\beta)\log |a|}.$$

Let $c(n)$ denote the coefficients in the expansion of f over over X. Then

$$\limsup_{n\to\infty} \frac{\log |c(n)|}{n^2} \le \beta \log |a|.$$

Proof: The estimate for $|c(n)|$ given in 4.1 implies that, for suitable constant C, if $r \ge 2|a|^n$ then

$$|c(n)| \le \frac{r}{r-|a|^n} M(f,r) \frac{|a^n-1|\cdots|a^n-a^{n-1}|}{(r-1)\cdots(r-|a|^{n-1})} \le CM(f,r)|a|^{n^2} r^{-n}.$$

Let b be chosen with $\beta < b < 1/2$, and set $r = |a|^{2(1-b)n}$. Then

$$\frac{\log M(f,r)}{(\log r)^2} \le \frac{1}{4(1-b)\log |a|}$$

once n is sufficiently large. Now

$$\frac{(\log r)^2}{4(1-b)\log|a|} = (1-b)n^2\log|a|$$

and

$$|c(n)| \le C|a|^{(1-b)n^2+n^2-2(1-b)n^2} = C|a|^{bn^2}.$$

Therefore

$$\limsup_{n\to\infty}\frac{\log|c(n)|}{n^2} \le b\log|a|.$$

Since this holds for any $b > \beta$, the conclusion follows. ∎

LEMMA 4.5: *Suppose $\alpha \in (-\infty, 1/2)$ and $c(n) \in \mathbb{C}, n \in \mathbb{N}$ satisfy*

$$\limsup_{n\to\infty}\frac{\log|c(n)|}{n^2} \le \alpha\log|a|$$

as $n \to \infty$. Then the series

$$f(z) = \sum_{n=0}^{\infty} c(n)\phi_{X,n}(z)$$

converges normally, thus determining an entire function $f(z)$, and $f(z)$ satisfies

$$\limsup_{r\to\infty}\frac{\log M(f,r)}{(\log r)^2} \le \frac{1}{4(1-\alpha)\log|a|}.$$

Proof: Let $\alpha < u < 1/2$. For suitable C (depending on u), $|c(n)| \le C|a|^{un^2}$. If $z \in \mathbb{C}$ it follows that

$$|c(n)\phi_{X,n}(z)| \le C|a|^{(u-1/2)n^2}(1+|z|)\cdots\left(1+\frac{|z|}{|a|^{n-1}}\right)$$

and normal convergence of the series follows.

Suppose now $r > 0$, and $|z| \le r$. Choose N such that $|a|^{N-1} \le r < |a|^N$. For $n \le N$ and $j \le n-1$, $1 + |z|/|a|^j \le |z|/|a|^{j-1}$ whence

$$|c(n)\phi_{X,n}(z)| \le C|a|^{(u-1/2)n^2+nN-n(n-1)/2+n} = C|a|^{(u-1)n^2+n(N+1/2)}.$$

The maximum of $(u-1)x^2 + (N+1/2)x$ occurs for $x = (N+1/2)/(2(1-u))$, hence

$$\sum_{n=0}^{N}|c(n)\phi_{X,n}(z)|$$

$$\le C\int_0^{N+1} e^{-(1-u)\log|a|x^2+(N+1/2)\log|a|x}dx + 2e^{\log|a|(N+1/2)^2/(4(1-u))}$$

$$\le \left(C\sqrt{(1-u)\log|a|2\pi}+2\right)e^{\log|a|(N+1/2)^2/(4(1-u))}$$

by extending the range of integration to $(-\infty, \infty)$. For $n > N$

$$|c(n)\phi_{X,n}(z)| \leq C\Pi^{+}|a|^{(u-1/2)n^2+N^2-n(n-1)/2+N+1},$$

so that

$$\sum_{n=N+1}^{\infty} |c(n)\phi_{X,n}(z)| \leq C\Pi^{+}|a|^{uN^2+N+1}\sum_{j=0}^{\infty}|a|^{2(u-1)Nj} = \frac{C\Pi^{+}|a|^{uN^2+N+1}}{1-|a|^{2(u-1)N}}.$$

Now $u \leq 1/(4(1-u))$, so the estimate of terms with $n \leq N$ dominates the estimate of terms with $n > N$. Since $\log r \geq (N-1)\log|a|$ it follows that

$$\limsup_{r\to\infty} \frac{\log M(f,r)}{(\log r)^2} \leq \frac{1}{4(1-u)\log|a|}.$$

Since this holds for any u with $\alpha < u < 1/2$, the conclusion follows. ∎

The following result establishes conditions under which f is represented by its expansion over X.

LEMMA 4.6: *Suppose f is entire and*

$$\frac{\log M(f,r)}{\log|a|} \leq \frac{1}{2}\Big(\frac{\log r}{\log|a|}\Big)^2 + \frac{1}{2}\Big(\frac{\log r}{\log|a|}\Big) - \omega(r)$$

where $\omega(r) \to \infty$ as $r \to \infty$. Then f is represented by its expansion over X. In particular, f is represented by its expansion over X if

$$\limsup_{r\to\infty} \frac{\log M(f,r)}{(\log r)^2} \leq \frac{1}{2\log|a|}.$$

Proof: The expansion takes the form $f(z) = \sum_{n=0}^{\infty} c(n)\phi_{X,n}(z)$ where

$$c(n) = \frac{1}{2\pi i}\int_C \frac{f(t)dt}{(t-x_n)\phi_{X,n}(t)},$$

the integral being over the circle C of radius $r > |a|^n$.

To establish the representation, let $z \in \mathbb{C}$ and for $n \in \mathbb{N}$ set

$$W_n(z) = f(z) - \sum_{j=0}^{n} c(j)\phi_j(z).$$

Then, as in [25, 4.6], $W_n(z)$ may be expressed as a contour integral over the circle C_n of radius $s_n > r_n = |a|^n$,

$$W_n(z) = \phi_{X,n}(z)\frac{1}{2\pi i}\int_C \frac{f(t)dt}{(t-z)\phi_{X,j}(t)}.$$

Choosing $s_n = |a|^{n+1/2}$ yields, for an appropriate choice of constant $C > 0$ (depending on a and z), the estimate

$$|W_n(z)| \leq CM(f, s_n)|a|^{n(n-1)/2-n(n+1/2)} \leq C|a|^{3/8-\omega(s_n)}.$$

Since $\omega(s_n) \to \infty$ as $n \to \infty$, $|W_n(z)| \to 0$, and the proof is complete. ∎

The leading coefficient 1/2 is best possible, in view of the entire function

$$H(z) = H_a(z) = \prod_{n=0}^{\infty} \left(1 - \frac{z}{a^n}\right)$$

which vanishes on X and satisfies $\limsup_{r\to\infty} \left(\log M(H, r)/(\log r)^2\right) = 1/2$.

5. Proof of the main theorems

Considering first the results for $X = \mathbb{N}$, the proof of Theorem 1.3 requires the following observation.

LEMMA 5.1: *Let $k \in \mathbb{N}$, $\gamma_k = \sum_{\kappa=1}^{k} 1/\kappa$. Then as $n \to \infty$,*

$$\lambda_{\mathbb{N}}(k, n) = e^{\gamma_k n + o(n)}.$$

Proof: This follows from the Prime Number Theorem. ∎

Proof of Theorem 1.3: Let f be entire of exponential type $b < \log(e^{\gamma_k} + 1)$, and $c(n)$ the coefficients in its expansion over $\mathbb{N}$. Choose β with $\log(e^b - 1) < \beta < \gamma_k$. Then by 4.2, for all sufficiently large n, $|c(n)| \leq e^{\beta n}$. However by 1.9, since f is concordant to order k, $c(n)$ is divisible by $\lambda_{\mathbb{N}}(k, n)$ for all n. Therefore, in view of 5.1, $c(n) = 0$ for all sufficiently large n. ∎

Proof of Theorem 1.4: Suppose f is superconcordant on $\mathbb{N}$. Then, taking $r = n + 1$ in the estimate of 4.1,

$$|c(n)| \leq \frac{n!M(f, r)}{(r-1)\cdots(r-n)} = \frac{n!M(f, r)}{\Gamma(r)}.$$

Under the assumptions on f,

$$\limsup_{n\to\infty} \frac{|c(n)|}{n!} < 1.$$

Since f is superconcordant, $c(n)$ is divisible by $n!$. Hence $c(n) = 0$ for all sufficiently large n. ∎

Turning to the results for X_a, the following result is needed in the proof of Theorem 1.2.

LEMMA 5.2: *Let* $a \in \mathbb{Z}$, $|a| \geq 2$, $X = X_a$, $k \in \mathbb{N}$, $\alpha_k = (3/\pi^2)\sum_{\kappa=1}^{k} \kappa^{-2}$. *Then*

$$\limsup_{n\to\infty} \frac{\log \lambda_X(k,n)}{n^2} = \alpha_k \log |a|.$$

Proof: By 2.5, $\lambda_X(k,n) = |b_X(k,n)|$. Write $b_X(k,n) = \prod_{\kappa=1}^{k} \beta(\kappa,n)$, where

$$\begin{aligned}\beta(\kappa,n) &= \prod_{r=1}^{[n/\kappa]} \prod_{s|r} (a^{n-\kappa r+r/s} - a^{n-\kappa r})^{\mu(s)} \\ &= \prod_{r=1}^{[n/\kappa]} \Big(\prod_{s|r} (a^{n-\kappa r})^{\mu(s)}\Big)\Big(\prod_{s|r} (a^{r/s}-1)^{\mu(s)}\Big).\end{aligned}$$

Now $\sum_{s|r} \mu(s) = 0$ unless $r = 1$, in which case $\sum_{s|r} \mu(s) = 1$. Therefore

$$\prod_{r} \prod_{s|r} (a^{n-\kappa r})^{\mu(s)} = a^{n-\kappa},$$

and so

$$\beta(\kappa,n) = a^{n-\kappa} \prod_{r=1}^{[n/\kappa]} \prod_{s|r} a^{r\mu(s)/s}\big(1-a^{-r/s}\big)^{\mu(s)}.$$

Thus $|\beta(n,\kappa)| = |a|^{e(n,\kappa)+O(n)}$ where the implied constant depends on $|a|, k$ and

$$e(n,\kappa) = n-\kappa + \sum_{r=1}^{[n/\kappa]} r \sum_{s|r} \frac{\mu(s)}{s} = n-\kappa+\sum_{r=1}^{[n/\kappa]} \phi(r)$$

where $\phi(j)$ is the Euler ϕ-function. Now by [19, Theorem 330]

$$\sum_{r=1}^{n} \phi(r) = \frac{3}{\pi^2}n^2 + O(n\log n).$$

Therefore

$$\sum_{\kappa=1}^{k} \sum_{r=1}^{[n/\kappa]} \phi(r) = \frac{3}{\pi^2} n^2 \sum_{\kappa=1}^{k} \frac{1}{k^2} + O(n \log n),$$

and this completes the proof. ∎

Proof of Theorem 1.2: Set

$$T_{a,k}(z) = \sum_{n=0}^{\infty} b_X(k,n)\phi_{X,n}(z).$$

By 5.2 and 4.5 it holds that $T_{a,k}$ is entire with

$$\limsup_{r\to\infty} \frac{\log M(T_{a,k}, r)}{(\log r)^2} = \frac{1}{4(1-\alpha_k)\log|a|}.$$

The growth estimate implies that $T_{a,k}$ is transcendental, which also follows immediately from the fact that the expansion coefficients $b_X(k,n)$ do not vanish for all sufficiently large n. Further, $T_{a,k}$ is concordant to order k on X by 1.9.

Now suppose f is concordant to order k on X. Let $c(n)$ denote the expansion coefficients of f over X. By 1.9, $c(n)$ is divisible by $\lambda_X(k,n)$ for each $n \in \mathbb{N}$. If infinitely many $c(n)$ are nonzero then, by 5.2,

$$\limsup_{n\to\infty} \frac{\log|c(n)|}{n^2} \geq \alpha_k \log|a|,$$

whence, by 4.4,

$$\limsup_{r\to\infty} \frac{\log M(f, r)}{(\log r)^2} \geq \frac{1}{4(1-\alpha_k)\log|a|}.$$

However, f is hypothesised to fail this condition. So only finitely many $c(n)$ are nonzero. By 4.6, f is represented by its expansion over X, hence f is a polynomial. ∎

Proof of Theorem 1.5: By 4.3 the expansion coefficients $c(n)$ of f satisfy

$$\limsup_{n\to\infty} \frac{\log|c(n)|}{n^2} < \alpha \log|a|$$

where $\alpha < 1$. Since f is superconcordant on X, $c(n)$ is divisible by $(a^n - 1)\cdots(a^n - a^{n-1})$ for each n by 1.9. Now

$$\lim_{n\to\infty} \frac{\log|(a^n-1)\cdots(a^n-a^{n-1})|}{n^2} = \log|a|.$$

Therefore $c(n) = 0$ for all sufficiently large n. ∎

In view of the canonical expressions provided by 2.4, it is natural to make the search for a smallest transcendental entire function $T_{\mathbb{N},k}$ concordant to order k on $\mathbb{N}$ more precise by requiring that it be of the minimal exponential type admitted by 1.3 with

$$T_{\mathbb{N},k}(n) = \sum_{j=0}^{n} b_X(k,n)\phi_{X,j}(n) = \sum_{j=0}^{n} b_X(k,j)\binom{n}{j}.$$

Turning to superconcordant functions, it is natural likewise to seek $T_{\mathbb{N},\infty}$ with, for $n \in \mathbb{N}$,

$$T_{\mathbb{N},\infty}(n) = \sum_{j=0}^{n} n(n-1)\cdots(n-j+1) = n! \sum_{j=0}^{n} \frac{1}{j!},$$

and

$$\lim_{r\to\infty} \frac{M(T_{\mathbb{N},\infty}, r)}{\Gamma(r+1)} = \sum_{j=0}^{\infty} \frac{1}{j!} = e$$

(in which case Theorem 1.4 might be sharpened).

Results on entire functions concordant to order $k \geq 2$ or superconcordant might also be pursued for the more general concordant subject sequences X of [2, 25]. The expansion series

$$T_{X,k}(z) = \sum_{j=0}^{n} b_X(k,n)\phi_{X,j}(z), T_{X,\infty}(z) = \sum_{j=0}^{n} \prod_{i=0}^{j-1} (z - x_i)$$

do not in general converge for $z \notin X$. Functions $T_{X,k}$, $T_{X,\infty}$ having the required values on X would necessarily be transcendental as, for any proper sequence $X \subset \mathbb{Z}$, the expansion coefficients of a polynomial are eventually zero.

Extending to higher orders of concordance the elegant interpolation of the (nonconvergent) series $\sum_{j=0}^{\infty} \phi_{\mathbb{N},j}(z)$ on the sequence $\mathbb{N}$ effected by the entire function $T_{\mathbb{N},0}(z) = 2^z$ for those sequences X for which the interpolation series does not itself determine an entire function thus presents an appealing problem. For the case $X = \mathbb{N}$ this problem is additionally intruiging as the values on $\mathbb{N}$ of the sought functions carry arithmetic information.

References

[1] E. Aichinger, *Local polynomial functions on the integers*, Rivista di Matematica della Università di Parma **6** (1997), 169–177.

[2] J.-P. Bézivin, *Itération de pôlynomes et fonctiones entières arithmétiques*, Acta Arithmetica **68** (1994), 11–25.

[3] J.-P. Bézivin, *Suites d'entières et fonctions entières arithmetiques*, Annales de la Faculté des Sciences de Toulouse Mathematics **3** (1994), 313–334.

[4] J.-P. Bézivin, *Sur les fonctions entières q-arithmétiques*, Rendiconti del Circolo Matematico di Palermo (2) **47** (1998), 447–462.

[5] R. P. Boas, *Comments on [27]*, in *George Pólya: Collected Papers* (R. P. Boas, ed.), Volume 1, MIT Press, Cambridge, 1974, pp. 771–773.

[6] R. C. Buck, *Integral valued entire functions,* Duke Mathematical Journal **15** (1948), 879–891.

[7] P. Bundschuh, *Arithmetische Eigenschaften ganzer Funktionen mehrer Variablen,* Journal für die reine und angewandte Mathematik **313** (1980), 116–132.

[8] P. Bundschuh and I. Shiokawa, *A remark on a theorem of Gel'fond,* Archiv der Mathematik (Basel) **65** (1995), 32–35.

[9] M. Car, *Pólya's theorem for $F_q[T]$,* Journal of Number Theory **66** (1997), 148–171.

[10] N. G. De Bruijn, *Some classes of integral-valued functions,* Indagationes Mathematicae **17** (1955), 363–367.

[11] G. A. Fridman, *Entire integer-valued functions,* Matematicheskii Sbornik (N.S.) **75 (117)** (1968), 417–431; English transl.: Mathematics of the USSR-Sbornik **4** (1968), 385–398.

[12] A. O. Gel'fond, *Sur les fonctions entière, que prenant des valuers entières dans les points β^n, β est un nombre entièr positif et $n = 1, 2, \ldots$.* (Russian), Matematicheskii Sbornik **40** (1933), 42–47.

[13] A. O. Gel'fond, *Calculus of Finite Differences* (authorised English translation of the third Russian edition), Hindustan Publishing Corporation, Delhi, 1971.

[14] A. O. Gel'fond, *Functions which take on integral values,* Mathematical Notes **1** (1967), 337–339; English translation of Matematicheskie Zametki **1** (1967), 509–513.

[15] A. O. Gel'fond, *Transcendental and Algebraic Numbers,* Chapter III, Translated from the first Russian edition by L. F. Boron, Dover, New York, 1960.

[16] F. Gramain, *Sur le théorèm de Fukasawa–Gel'fond,* Inventiones Mathematicae **63** (1981), 495–506.

[17] R. R. Hall, *On pseudopolynomials,* Mathematika **8** (1971), 71–77.

[18] G. H. Hardy, *On a theorem of Mr. G. Pólya,* Proceedings of the Cambridge Philosophical Society **19** (1917), 60–63.

[19] G. H. Hardy and Wright, *An Introduction to the Theory of Numbers,* Fifth edition, Clarendon Press, Oxford, 1979.

[20] W. Narkiewicz, *Polynomial Mappings,* Lecture Notes in Mathematics **1600**, Springer-Verlag, Berlin, 1995.

[21] W. Nöbauer, *Compatible and conservative functions on residue-class rings of the integers,* Colloquia Mathematica Societatis János Bolyai **13**, Topics in Number Theory, North-Holland, Debrecen, 1976, pp. 245–257.

[22] A. Perelli and U. Zannier, *Su un teorema di Polya,* Unione Matematica Italiana. Bollettino A **5** 18 (1981), 305–307.

[23] A. Perelli and U. Zannier, *On recurrent mod p sequences*, Journal für die reine und angewandte Mathematik **384** (1984), 135–146.

[24] J. Pila, *Concordant sequences and concordant entire functions*, Rendiconti del Circolo Matematico di Palermo (2) **51** (2002), 51–82.

[25] J. Pila and F. Rodriguez Villegas, *Concordant sequences and integral-valued entire functions*, Acta Arithmetica **88** (1999), 239–268.

[26] C. Pisot, *Über ganzwertige ganze Funktionen*, Jahresbericht der Deutscher Mathematiker Vereinigung **52** (1942), 95–102; Mathematical Review **4**, p. 270.

[27] G. Pólya, *Ueber ganzwertige ganze Funktionen*, Rendiconti del Circolo Matematico di Palermo **40** (1915), 1–16. Also in *Collected Papers* (R. P. Boas, ed.), Volume 1, MIT Press, Cambridge, 1974, pp. 1–16.

[28] G. Pólya, *Über ganz ganzwertige Funktionen*, Nachrichten Gesammelten Wissenschaften in Göttingen (1920), 1–10. Also in *Collected Papers* (R. P. Boas, ed.), Volume 1, MIT Press, Cambridge, 1974, pp. 131–140.

[29] R. M. Robinson, *Integer-valued entire functions*, Transactions of the American Mathematical Society **153** (1971), 451–468.

[30] I. Z. Ruzsa, *On congruence preserving functions*, Matematikai Lapok **22** (1971), 125–134.

[31] A. Selberg, *Über ganzwertige ganze transzendente Funktionen*, Archiv for Mathematik og Naturvidenskab B **44** (1941), 45–52. Also in *Collected Papers*, Volume 1, Springer-Verlag, Berlin, 1989, pp. 54–61.

[32] A. Selberg, *Über ganzwertige ganze transzendente Funktionen, II*, Archiv for Mathematik og Naturvidenskab B **44** (1941), 171–181. Also in *Collected Papers*, Volume 1, Springer-Verlag, Berlin, 1989, pp. 74–84.

[33] M. Welter, *On entire functions whose derivatives are integer-valued on geometric progressions*, Manuscripta Mathematica **102** (2000), 63–74.

[34] U. Zannier, *On periodic mod p sequences and G-functions*, Manuscripta Mathematica **90** (1996), 391–402.